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Market selection with learning and catching up with the Joneses

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Abstract We study the market selection hypothesis in complete financial markets, populated by heterogeneous agents. We allow for a rich structure of heterogeneity: individuals may differ in their beliefs concerning the economy, information and learning mechanism, risk aversion, impatience and 'catching up with the Joneses' preferences. We develop new techniques for studying the long-run behavior of such economies, based on Strassen's functional law of the iterated logarithm. In particular, we explicitly determine an agent's survival index and show how the latter depends on the agent's characteristics. We use these results to study the long-run behavior of the equilibrium interest rate and the market price of risk.

Keywords Natural selection · Heterogeneous equilibrium · Diverse beliefs · Learning · Survival index · Catching up with the Joneses

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1 Introduction

A fundamental question in the modern theory of financial economics is concerned with the so-called market selection hypothesis, dating back to the ideas of Friedman [16]. Motivated by the postulate that agents with inaccurate forecasts will eventually be driven out of the economy, this hypothesis can be stated informally as "If you

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are so smart, why aren't you rich?". Formally, market selection in financial markets examines the agents' long-run survival[1] capability and price impact in equilibrium models. There is a vast body of literature dealing with this topic; see e.g. Blume and Easley [7], Cvitanić et al. [11], Nishide and Rogers [25], Sandroni [27], and Yan [32, 33].

This paper investigates the market selection hypothesis (or natural selection, for short) and the long-run behavior of asset prices in a complete market setting with highly heterogeneous investors. Individuals may differ in their beliefs concerning the economy, information and learning mechanism, risk aversion, impatience (time preference rate) and degree of habit formation. Each individual in our model is represented by a generalized version of the *catching up with the Joneses* power utility function of Chan and Kogan [8]. This model of preferences is sometimes referred to in the literature as *exogenous habit formation*, since it incorporates the impact of a certain given stochastic process on the individual's consumption policy. Agents are assumed to possess only partial information regarding the events associated with the evolution of the market. More precisely, the stochastic dynamics of the mean growth rate of the economy[2] are *unobservable*, and the agents' information set consists of the aggregate endowment and a publicly observable signal. Furthermore, agents are allowed to have *diverse beliefs* concerning the values of the initial and average mean growth rate. Individuals may be irrational in the way they interpret the public signal: some of them may be over- (or under-)confident about the informativeness of the public signal. We use the standard way of modeling over- (or under-)confidence, originated in Dumas et al. [14] and Scheinkman and Xiong [28]: we assume that agents' beliefs concerning the instantaneous correlation of the public signal with the economy's growth rate may differ from its actual value.[3] The agents are rational in the sense that they use a standard Kalman filter to update their expectations about the economy's growth rate. The heterogeneous filtering rules yield highly nontrivial dynamics for the individual consumption and the equilibrium state price density, determined by the market clearing condition. In particular, subjective probability densities describing the agents' beliefs give rise to multiple new state variables, which govern the dynamics of the economy. We refer to Back [2] for a survey on filtering and incomplete information in asset pricing theory.

Let us describe the contribution of this work to the literature on equilibrium and natural selection. Firstly, as described above, we analyze a very general paradigm of heterogeneous economies including diverse beliefs, Kalman filtering and exogenous state-dependent habit formation preferences. We provide a comprehensive description of the equilibrium characteristics that can be used for further research in other possible directions. Secondly, this complex setting in turn allows detecting which traits (both behavioral-preferential and information-related) are beneficial for survival. That is, as in Yan [32], we reveal that there is a unique surviving agent in the

[1] An agent is said to survive in the long run if the ratio of his consumption to the aggregate consumption stays positive with positive probability as time goes to infinity.

[2] We assume that the mean growth rate follows an Ornstein–Uhlenbeck process.

[3] This is a realistic assumption as correlations are extremely difficult to estimate empirically.

long run. Moreover, we show that the interest rate and the market price of risk behave asymptotically as those of an economy populated solely by this surviving agent. Lastly, to derive our results, we develop new techniques based mainly on Strassen's functional law of the iterated logarithm. To the best of our knowledge, these methods have never been used in the general equilibrium literature before.

The conclusions and implications on natural selection are as follows. Most importantly, our findings indeed confirm, to a large extent, the validity of the market selection hypothesis.

In a growing economy, the less effectively risk-averse[4] agent is the one to survive in the long run. This result is consistent with previous studies (see e.g. Cvitanić et al. [11]). However, the impact of habit formation on the effective risk aversion, and thus in particular on survival, is quite novel. As it turns out, if the (standard non-effective) risk aversion coefficient is above one, then the individual with the strongest habit will survive. Intuitively, this makes sense, as aggressiveness in a growing economy among somewhat moderate individuals is supposed to be a plus. On the other hand, if the (standard non-effective) level of risk aversion is below one (i.e., individuals are relatively risk-seeking in the classical sense), the agent with the lowest degree of habit formation will dominate. This is not surprising at all, as excess aggressiveness can cause bubbles leading to extinction.

Some of our conclusions concerning the interaction of diverse beliefs and survival are quite intriguing, and seem to be quite hard to predict without a delicate analysis.

When agents differ only in their beliefs concerning the average mean growth rate, the one with the most accurate forecast will dominate the market, as expected. If all agents are over-confident (or under-confident), then, again, the agent with the best guess will beat the others. However, if some agents are over-confident and others are under-confident, the situation is more complex. For instance, it may happen that in a situation where the public signal provides some relevant information about the market, the surviving agent will be the one who (wrongly) believes that this signal is a pure noise, whereas the agent who is significantly over-confident in the informativeness of the signal will be eliminated from the economy. Furthermore, in some cases, agents that believe in a negative correlation of the signal will survive, while individuals who believe in a (too high) positive correlation will become extinct, despite an actual positive correlation. See Fig. 1 for an example describing these phenomena. Even though it is somewhat debatable which property of the preceding two can be considered a more rational one, we still learn that theoretically, the market selection hypothesis is valid, at least in some modified form.

We now review some related works. The most closely related to ours are the papers by Yan [32] and Cvitanić et al. [11].[5] Specifically, these authors consider a special case of our model corresponding to the case when there is no learning and agents have standard CRRA preferences without any habit formation. In terms of modeling heterogeneous beliefs and learning, our model closely follows the one of Dumas et al. [14] and Scheinkman and Xiong [28], who considered a special case of our model:

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[4]In our model, the effective risk aversion depends on the level of habit formation (see (4.1)).

[5]Bhamra and Uppal [5], Dumas [13] and Wang [29] considered the same model, but with only two agent types and heterogeneity coming only from risk aversion.

a two-agent economy with standard CRRA utility functions, and the public signal being a pure noise, uncorrelated with the economy's growth rate. Chan and Kogan [8] consider a special case of our model with homogeneous 'catching up with the Joneses' habit levels and a continuum of agents with heterogeneous risk aversions. Xiouros and Zapatero [31] derive a closed form expression for the equilibrium state price density in the Chan and Kogan [8] model. Cvitanić and Malamud [9] study how long-run risk sharing depends on the presence of multiple agents with different levels of risk aversion. Kogan et al. [21] and Cvitanić and Malamud [10] study the interaction of survival and price impact in economies where agents derive utility only from terminal consumption. Fedyk et al. [15] extend the model of Yan [32] by allowing for many assets. Kogan et al. [22] study the link between survival and price impact in the presence of intermediate consumption, and allow for general utilities with unbounded relative risk aversion and a general dividend process. Another quite significant direction of the complete market risk sharing literature concentrates on the equilibrium effects of heterogeneous beliefs. Bhamra and Uppal [6] derive a characterization of the equilibrium state price density by means of an infinite series that admits a closed form solution for specific coefficients, in a two-agent economy with diverse beliefs and heterogenous CRRA preferences. With CRRA agents differing only in their beliefs, the equilibrium state price density can be derived in a closed form, and thus many equilibrium properties can be analyzed in detail. See e.g. Basak [3, 4], Jouini and Napp [18, 19], Jouini et al. [20] and Xiong and Yan [30].

The paper is organized as follows. In Sect. 2, we introduce the model and provide some preliminary results. Section 3 is devoted to a brief description of the equilibrium state price density in homogeneous and heterogeneous settings. In Sect. 4, we present the main result of the paper and discuss some implications. Section 5 deals with some auxiliary results that are crucial for the proof of the main result. In Sect. 6, we prove the main result. Finally, in Sect. 7 we establish long-run results for the interest rate and the market price of risk. Some of the results appearing in Sects. 5 and 7 are of independent mathematical interest.

2 Preliminaries

We consider a continuous-time Arrow–Debreu economy with an infinite horizon in which heterogeneous agents maximize their utility functions from consumption. The uncertainty in our model is captured by a (complete) probability space $(\Omega, \mathcal{F}_\infty, P)$ and a continuous filtration $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,\infty)}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We fix three standard and independent Wiener processes $(W_t^{(i)})_{t\in[0,\infty)}$, $i = 1, 2, 3$, adapted to the filtration $\mathcal{F}$. There are N different types of agent in the economy, labeled by $i = 1, \ldots, N$. Each agent i is equipped with a nonnegative endowment process $(\epsilon_t^i)_{t\in[0,\infty)}$ adapted to the filtration $\mathcal{G}$ (see (2.5)). We denote by $D_t := \sum_{i=1}^N \epsilon_t^i$ the *aggregate endowment* process and assume that $(D_t)_{t\in[0,\infty)}$ satisfies

$$\frac{dD_t}{D_t} = \mu_t^D\, dt + \sigma^D\, dW_t^{(1)}, \qquad D_0 = 1,$$

or equivalently

$$D_t = \exp\biggl(\int_0^t \mu_s^D \, ds - \frac{1}{2}(\sigma^D)^2 t + \sigma^D W_t^{(1)}\biggr), \tag{2.1}$$

where the constant $\sigma^D > 0$ represents the volatility. The *mean growth rate* $(\mu_t^D)_{t\in[0,\infty)}$ is an Ornstein–Uhlenbeck process that solves uniquely the SDE

$$d\mu_t^D = -\xi\left(\mu_t^D - \overline{\mu}\right) dt + \sigma^\mu \, dW_t^{(2)}, \qquad \mu_0^D = \mu,$$

that is,

$$\mu_t^D = \overline{\mu} + (\mu_0 - \overline{\mu})e^{-\xi t} + \sigma^\mu e^{-\xi t} \int_0^t e^{\xi s} \, dW_s^{(2)}, \tag{2.2}$$

where $\overline{\mu}$, μ_0 and σ^μ are some real numbers and $\xi > 0$. The numbers $\overline{\mu}$, μ_0 will be referred to as the *average* and *initial* mean growth rate, respectively.

2.1 The financial market

We consider a financial market that consists of at least two long-lived securities: a risky stock $(S_t)_{t\in[0,\infty)}$ and a bank account $(S_t^0)_{t\in[0,\infty)}$. In addition to this, there are other (not explicitly modeled) assets guaranteeing that the market is dynamically complete[6] for $\mathcal{G}$-adapted claims (where the filtration $\mathcal{G} := (\mathcal{G}_t)_{t\in[0,\infty)}$ is defined in (2.5)). We emphasize that this filtration coincides with the symmetric information shared by all agents. The bond is in zero net supply and the stock is a claim to the total endowment of the economy $(D_t)_{t\in[0,\infty)}$ and has a net supply of one share. The riskless bond is given by $S_t^0 = e^{\int_0^t r_s \, ds}$, where $(r_t)_{t\in[0,\infty)}$ is the riskfree rate process. We assume a *unique positive state price density* denoted by $(M_t)_{t\in[0,\infty)}$, that is, a positive process adapted to $\mathcal{G}$ that satisfies

$$M_t = E\bigl[e^{\int_t^u r_s \, ds} M_u \big| \mathcal{G}_t\bigr]$$

for all $u > t$, and

$$S_t = E\biggl[\int_t^\infty \frac{M_u}{M_t} D_u \, du \bigg| \mathcal{G}_t\biggr]$$

for all $t > 0$. Note that our assumption excludes arbitrage opportunities in the model. The state price density, as well as all other parameters, are to be derived endogenously in equilibrium.

[6] In this setting, the model can be implemented by a complete securities market with a unique state price density derived in equilibrium (as for instance in Duffie and Huang [12]). More specifically, the filtration $\mathcal{G}$ is generated by the Brownian motion s (which is interpreted as a public signal) and the aggregate endowment process D. Nevertheless, as explained in Remark 2.6, the filtration $\mathcal{G}$ is also generated by the Brownian motions s and $W^{(0)}$. Thereby, the market can be completed by adding one additional security to S. However, since the price of this security would be determined endogenously, one would have to verify endogenous completeness. This can be done by using the techniques of Hugonnier et al. [17]. Otherwise, we can just assume that there are sufficiently many (derivative) assets, completing the market.

2.2 Preferences and equilibrium

Agent i is maximizing his intertemporal von Neumann–Morgenstern expected utility,

$$\sup_{(c_{it})_{t\in[0,\infty)}} E^{P^i}\left[\int_0^\infty e^{-\rho_i t} U_i(c_{it})\,dt\right],$$

from consumption, under the constraints that the consumption stream $(c_{it})_{t\in[0,\infty)}$ is a positive process adapted to $\mathcal{G}$ (which is defined in (2.5)) and lies in the budget set,

$$E\left[\int_0^\infty c_{it} M_t\,dt\right] \le E\left[\int_0^\infty \epsilon_t^i M_t\,dt\right].$$

Here, $E^{P_i}[\cdot]$ stands for the expectation with respect to the subjective probability measure P_i of agent i. The exact form of P_i is specified in (2.13). We assume that all agents are represented by 'catching up with the Joneses'[7] preferences

$$U_i(c_{it}) = \frac{1}{1-\gamma_i}\left(\frac{c_{it}}{H_{it}}\right)^{1-\gamma_i}.$$

The subjective 'standard of living' index $(H_{it})_{t\in[0,\infty)}$ is defined through a certain geometric average of the aggregate endowment process. We consider here a more general specification for H_{it} than the one in Chan and Kogan [8]. Namely, we set $H_{it} = e^{\beta_i x_t}$ for some $\beta_i \ge 0$, where

$$x_t = e^{-\lambda t}\left(x_0 + \lambda\int_0^t e^{\lambda s}\log D_s\,ds\right), \tag{2.3}$$

or, equivalently, $(x_t)_{t\in[0,\infty)}$ solves the SDE

$$dx_t = \lambda(\log D_t - x_t)\,dt.$$

For each agent i, the number β_i measures the impact of the index x_t on the agent; in particular, when $\beta_i = 0$, the agent is not influenced by the index at all. For large β_i, the influence is somewhat heavy. In complete markets, the optimal consumption stream can be easily derived as in the following statement.

Proposition 2.1 *The optimal consumption stream of agent i in a complete market represented by a state price density $(M_t)_{t\in[0,\infty)}$ is given by*

$$c_{it} = e^{\frac{\rho_i}{\gamma_i}t} M_t^{-\frac{1}{\gamma_i}} Z_{it}^{\frac{1}{\gamma_i}} H_{it}^{\frac{\gamma_i-1}{\gamma_i}} c_{i0},$$

[7] This paradigm of a utility function was first introduced in Abel [1], and is commonly referred to in the literature as a utility with exogenous habits. This specification describes a decision maker who experiences an impact of the 'standard of living' index.

and

$$E\left[\int_0^\infty c_{it}\xi_t\right] = E\left[\int_0^\infty \epsilon_t^i M_t\right],$$

where the density process $(Z_{it})_{t\in[0,\infty)}$ *is given in* (2.13).

Proof The assertion follows by standard duality arguments involving the first-order conditions. □

Finally, we introduce the notion of Arrow–Debreu equilibrium.

Definition 2.2 An *equilibrium* is a pair $((c_{it})_{t\in[0,\infty)}, (M_t)_{t\in[0,\infty)})$ such that

(a) Each process $(c_{it})_{t\in[0,\infty)}$ is the optimal consumption stream of agent i, and $(M_t)_{t\in[0,\infty)}$ is the state price density that represents the market.
(b) The market clearing condition is satisfied, i.e.,

$$\sum_{i=1}^N c_{it} = D_t \tag{2.4}$$

for all $t > 0$.

2.3 Diverse beliefs and learning

There are two processes in the economy that are *observable* by all agents. The first one is the aggregate endowment process $(D_t)_{t\in[0,\infty)}$, and the second one is a certain public signal

$$s_t = \phi W_t^{(2)} + \sqrt{1-\phi^2}\, W_t^{(3)},$$

for some $\phi \in [0, 1)$. That is, the public signal exhibits a nonnegative correlation $\phi \in [0, 1)$ with the shock governing the mean growth rate process. The corresponding filtration is denoted by

$$\mathcal{G}_t := \sigma\big(\{s_u; u \le t\} \cup \{D_u; u \le t\}\big). \tag{2.5}$$

In contrast to this, the mean growth rate process is *unobservable*. That is, neither of the agents possesses access to the data revealing the dynamics of the process $(\mu_t^D)_{t\in[0,\infty)}$. Furthermore, agents may have diverse beliefs concerning the average and initial mean growth rate. More precisely, each agent i believes that the initial mean growth rate is some $\mu_{0i} \in \mathbb{R}$ and that the average mean growth rate is some $\overline{\mu}_i \in \mathbb{R}$. That is to say, before filtering, agent i assigns in his mind for μ_t^D the model

$$\overline{\mu}_i + (\mu_{i0} - \overline{\mu}_i)e^{-\xi t} + \sigma^\mu e^{-\xi t}\int_0^t e^{\xi s}\, dW_s^{(2)}. \tag{2.6}$$

Furthermore, individuals may have an irrational perception of the signal. Concretely, each agent i believes that the public signal $(s_t)_{t\in[0,\infty)}$ has a correlation $\phi_i \in [-1, 1)$

with $(W_t^{(2)})_{t\in[0,\infty)}$, when in fact the correlation is $\phi \in [0,1)$. Therefore, under the belief of agent i, the model attributed to the signal s_t is

$$\phi_i W_t^{(2)} + \sqrt{1-\phi_i^2}\, W_t^{(3)}. \tag{2.7}$$

We denote by Q^i the measure corresponding to agent i's beliefs regarding the models in (2.6) and (2.7), where $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are independent Wiener processes under Q^i. Consequently, agents are in the process of learning and filtering out the dynamics of the mean growth rate, which is deduced by using the theory of optimal filtering.

Definition 2.3 The process

$$\mu_{it}^D := E^{Q^i}\left[\overline{\mu}_i + (\mu_{i0} - \overline{\mu}_i)e^{-\xi t} + \sigma^\mu e^{-\xi t}\int_0^t e^{\xi s}\, dW_s^{(2)}\middle|\mathcal{G}_t\right]$$

is called the *subjective mean growth rate* of agent i.

Proposition 2.4 *We have*

$$\mu_{it}^D = \frac{\mu_{i0}}{y_{it}} + \frac{\xi\overline{\mu}_i}{y_{it}}\int_0^t y_{iu}\, du + \frac{1}{(\sigma^D)^2 y_{it}}\int_0^t \frac{\nu_{iu} y_{iu}}{D_u}\, dD_u + \frac{\sigma^\mu\phi_i}{y_{it}}\int_0^t y_{iu}\, ds_u, \tag{2.8}$$

where

$$y_{it} = \exp\left(\xi t + \frac{1}{(\sigma^D)^2}\int_0^t \nu_{is}\, ds\right), \tag{2.9}$$

and the variance process

$$\nu_{it} := E^{Q^i}\left[\left(\mu_t^D - E^{Q^i}\left[\mu_t^D|\mathcal{G}_t\right]\right)^2\middle|\mathcal{G}_t\right]$$

is deterministic and given by

$$\nu_{it} = \alpha_{i2}\left(\sigma^D\right)^2 \frac{e^{(\alpha_{i2}-\alpha_{i1})t} - 1}{e^{(\alpha_{i2}-\alpha_{i1})t} - \alpha_{i2}/\alpha_{i1}}, \tag{2.10}$$

where

$$\alpha_{i2} = \sqrt{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi_i^2\right)} - \xi,$$

and

$$\alpha_{i1} = -\sqrt{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi_i^2\right)} - \xi.$$

Proof Observe that Theorem 12.7 in Liptser and Shiryaev [24] implies that $(\mu_{it}^D)_{t\in[0,\infty)}$ satisfies the SDE

$$d\mu_{it}^D = -\xi\left(\mu_{it}^D - \overline{\mu}_i\right)dt + \frac{\nu_{it}}{(\sigma^D)^2}\left(\frac{dD_t}{D_t} - \mu_{it}^D\, dt\right) + \sigma^\mu\phi_i\, ds_t, \tag{2.11}$$

where the variance process ν_{it} is detected through the Riccati ODE

$$\nu'_{it} = -2\xi\nu_{it} + (\sigma_\mu)^2\big(1-\phi_i^2\big) - \frac{1}{(\sigma^D)^2}\nu_{it}^2,$$

with $\nu_{i0}=0$. One can solve the above equation and verify that ν_{it} is given by (2.10). Now, we shall solve the SDE (2.11). By definition, we have $y'_{it} = (\xi + \frac{\nu_{it}}{(\sigma^D)^2})y_{it}$, and $y_{i0}=1$. Notice that the preceding observation combined with Itô's formula implies that

$$d\big(y_{it}\mu_{it}^D\big) = \xi\overline{\mu}_i y_{it}\,dt + \frac{\nu_{it}}{(\sigma^D)^2}y_{it}\frac{dD_t}{D_t} + \sigma^\mu\phi_i y_{it}\,ds_t,$$

completing the proof. □

Remark 2.5 Dumas et al. [14] consider the static version of (2.8). That is, the functions ν_{it} and y_{it} are substituted by the corresponding asymptotic limits. This can be justified by Lemma 5.1 of the current paper.

We denote by $i=0$ a fictional agent who is rational in the sense that he knows the correct average, initial mean growth rate and the correlation parameter ϕ. Let us denote by $\mu_{0t}^D := E^P[\mu_t^D|\mathcal{G}_t]$ the estimated mean growth rate of this agent. As in Proposition 2.4, we have

$$\mu_{0t}^D = \frac{\mu_0}{y_{0t}} + \frac{\xi\overline{\mu}}{y_{0t}}\int_0^t y_{0u}\,du + \frac{1}{(\sigma^D)^2 y_{0t}}\int_0^t \frac{\nu_{0u}y_{0u}}{D_u}\,dD_u + \frac{\sigma^\mu\phi}{y_{0t}}\int_0^t y_{0u}\,ds_u,$$

where y_{0t} and ν_{0t} are defined similarly to (2.9) and (2.10). It can be shown as in Theorem 8.1 in Liptser and Shiryaev [23] that $W_t^{(0)} = W_t^{(1)} - \int_0^t \frac{\mu_{0s}^D - \mu_s^D}{\sigma^D}\,ds$ is a P-Brownian motion with respect to the filtration $\mathcal{G}$.

Remark 2.6 The filtration $\mathcal{G}$ is generated by the public signal s and the Brownian motion $W^{(0)}$. To see this, note that

$$\frac{dD_t}{D_t} = \mu_{0t}^D\,dt + \sigma^D\,dW_t^{(0)},$$

and

$$d\mu_{0t}^D = -\xi\big(\mu_{0t}^D - \overline{\mu}\big)\,dt + \frac{\nu_{0t}}{\sigma^D}\,dW_t^{(0)} + \sigma^\mu\phi\,ds_t.$$

We set

$$\delta_{it} := \frac{\mu_{it}^D - \mu_{0t}^D}{\sigma^D} \tag{2.12}$$

to be the ith agent's *error* in the mean growth rate estimation. The dynamics of $(D_t)_{t\in[0,\infty)}$ from the ith agent's perspective admit the form

$$\frac{dD_t}{D_t} = \mu_{it}^D\,dt + \sigma^D\,dW_{it}^{(0)},$$

where

$$dW_{it}^{(0)} = dW_t^{(0)} - \delta_{it}\, dt$$

is a Brownian motion (by Girsanov's theorem) under the equivalent probability measure[8] P_i and the filtration $\mathcal{G}$, where

$$Z_{it} := E\left[\frac{dP^i}{dP}\bigg|\mathcal{G}_t\right] = \exp\left(\int_0^t \delta_{is}\, dW_s^{(0)} - \frac{1}{2}\int_0^t \delta_{is}^2\, ds\right). \tag{2.13}$$

Let us stress that $W_i^{(0)}$ is also a Q^i-Brownian motion with respect to the filtration $\mathcal{G}$. In particular, this implies that by restricting the measure Q^i to the sigma-algebra generated by $W_i^{(0)}$, we get the measure P^i. Nevertheless, the measures Q^i and P (the physical probability measure) are singular on the sigma-algebra (see (2.6) and (2.7)) generated by the Brownian motions $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$.

3 The equilibrium state price density

In the current section, we depict the structure of the equilibrium state price density (SPD) in both settings of homogeneous and heterogeneous economies.

3.1 Homogeneous economy

Consider an economy where all agents are of the same type i, and denote by $(M_{it})_{t\in[0,\infty)}$ the corresponding equilibrium state price density. The homogeneity of the economy combined with the completeness of the market allows us to derive the corresponding state price density in a closed form.

Lemma 3.1 *The equilibrium state price density in a market populated by one agent of type i is given by*

$$\begin{aligned} M_{it} &= e^{-\rho_i t} D_t^{-\gamma_i} Z_{it} H_{it}^{\gamma_i - 1} \\ &= \exp\left(-\int_0^t \left(\rho_i + \gamma_i\left(\mu_{0s}^D - \frac{1}{2}(\sigma^D)^2\right) + \frac{1}{2}\delta_{is}^2\right) ds\right) \\ &\quad \times \exp\left((\gamma_i - 1)\beta_i x_t + \int_0^t \left(\delta_{is} - \gamma_i \sigma^D\right) dW_s^{(0)}\right). \end{aligned} \tag{3.1}$$

Proof The assertion follows by using the market clearing condition and Lemma 2.1. □

We derive next the riskfree rate and the market price of risk in a homogeneous economy.

[8] One can check that the process $(Z_{it})_{t\in[0,\infty)}$ is a true martingale by verifying Novikov's condition on a small interval and then applying a similar argument to the one used in Example 3 in Sect. 6.2 in Liptser and Shiryaev [23].

Lemma 3.2 *The riskfree rate and the market price of risk in an economy populated by one agent of type i are given, respectively, by*

$$r_{it} := \rho_i + \gamma_i \mu_{it}^D - \frac{1}{2}\left(\sigma^D\right)^2 \gamma_i(\gamma_i + 1) - \beta_i(\gamma_i - 1)(x_t - \log D_t)$$

and

$$\theta_{it} := \gamma_i \sigma^D - \delta_{it}.$$

Proof Consider the process

$$Y_{it} := -\int_0^t \left(\rho_i + \gamma_i\left(\mu_{0s}^D - \frac{1}{2}\left(\sigma^D\right)^2\right) + \frac{1}{2}\delta_{is}^2\right) ds + (\gamma_i - 1)\beta_i x_t$$
$$+ \int_0^t \left(\delta_{is} - \gamma_i \sigma^D\right) dW_s^{(0)}.$$

The dynamics of M_i are given by

$$\frac{dM_{it}}{M_{it}} = dY_{it} + \frac{1}{2} d\langle Y_i, Y_i\rangle_t,$$

where

$$dY_{it} = -\left(\rho_i + \gamma_i\left(\mu_t^D - \frac{1}{2}\left(\sigma^D\right)^2\right) + \frac{1}{2}\delta_{it}^2\right) dt$$
$$+ \beta_i(\gamma_i - 1)(\log D_t - x_t)\, dt + \left(\delta_{it} - \gamma_i \sigma^D\right) dW_t^{(0)},$$

and

$$d\langle Y_i, Y_i\rangle_t = \left(\delta_{it} - \gamma_i \sigma^D\right)^2 dt.$$

The rest of the proof follows from the fact that the riskfree rate and the market price of risk coincide with minus the drift and minus the volatility of the SPD, respectively. □

3.2 Heterogeneous economy

Consider an economy populated by N different types of agent. By Lemma 2.1, the optimal consumption stream of agent i is given by

$$c_{it} = e^{-\frac{\rho_i}{\gamma_i} t} M_t^{-\frac{1}{\gamma_i}} Z_{it}^{\frac{1}{\gamma_i}} H_{it}^{\frac{\gamma_i - 1}{\gamma_i}} c_{i0} = c_{i0}\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} D_t, \tag{3.2}$$

where $(M_t)_{t\in[0,\infty)}$ stands for the corresponding heterogeneous equilibrium state price density, and M_{it} is given by (3.1). Therefore, the market clearing condition (2.4) admits the form

$$\sum_{i=1}^N c_{i0}\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} = 1. \tag{3.3}$$

Example 3.3 Consider a homogeneous risk aversion economy where we have $\gamma_1 = \cdots = \gamma_N = \gamma$. Then the equilibrium state price density is given explicitly by

$$M_t = \left(\sum_{i=1}^{N} \frac{c_{i0} e^{-\rho_i t/\gamma} Z_{it}^{1/\gamma} H_{it}^{\frac{\gamma-1}{\gamma}}}{D_t} \right)^{\gamma}.$$

Furthermore, if the habits are homogeneous, that is, $\beta_1 = \cdots = \beta_N = \beta$, we have

$$M_t = e^{(\gamma-1)\beta x_t} \left(\sum_{i=1}^{N} \frac{c_{i0} e^{\rho_i t/\gamma} Z_{it}^{1/\gamma}}{D_t} \right)^{\gamma}.$$

If the beliefs among the agents are not varying, i.e., $Z_{1t} = \cdots = Z_{Nt} = Z_t$, then we have

$$M_t = Z_t \left(\sum_{i=1}^{N} \frac{c_{i0} e^{\rho_i t/\gamma} H_{it}^{\frac{\gamma-1}{\gamma}}}{D_t} \right)^{\gamma}.$$

Finally, we provide formulas for the riskfree rate and the market price of risk.

Proposition 3.4 *We have*

$$\theta_t = \sum_{i=1}^{N} \omega_{it} \theta_{it}$$

and

$$r_t = \sum_{i=1}^{N} \omega_{it} r_{it} + \frac{1}{2} \sum_{i=1}^{N} (1 - 1/\gamma_i) \omega_{it} (\theta_{it} - \theta_t)^2,$$

where

$$\omega_{it} := \frac{c_{it}/\gamma_i}{\sum_{j=1}^{N} c_{jt}/\gamma_j}$$

denotes the relative level of absolute risk tolerance of agent i.

Proof The proof is identical to the one of Proposition 4.1 in Cvitanić et al. [11]. □

4 The main result: the long-run surviving consumer

The current section is devoted to the study of the long-run behavior of the optimal consumption shares in a heterogeneous economy. We establish the existence of a surviving consumer in the market, i.e., an agent whose optimal consumption asymptotically behaves as the aggregate consumption. This dominating individual is determined through the *survival index*. The survival index is a quantity depending on individuals' characteristics and specifies the surviving agent versus the agents to become extinct in the economy.

Definition 4.1 The *survival index* of agent i is given by

$$\kappa_i := \rho_i + \left(\overline{\mu} - \frac{1}{2}(\sigma^D)^2\right)\left(\gamma_i + (1-\gamma_i)\beta_i\right) + \frac{1}{2}\left(\frac{\overline{\mu}_i - \overline{\mu}}{\sigma^D}\right)^2 + \frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}}.$$

The following is assumed throughout the entire paper.

Assumption There exists an agent I_K whose survival index is the lowest one, namely $\kappa_{I_K} < \kappa_i$, for all $i \neq I_K$.

We are now ready to state our main result.

Theorem 4.2 *In equilibrium, the only surviving agent in the long run is the one with the lowest survival index, i.e.,*

$$\lim_{t\to\infty} \frac{c_{it}}{D_t} = 0$$

for all $i \neq I_K$, and

$$\lim_{t\to\infty} \frac{c_{I_K t}}{D_t} = 1.$$

The survival index is a complicated function of the individuals' underlying parameters. In order to isolate the effects of various agents' characteristics on the long-run survival, we discuss special cases in which agents differ with respect to only one or a few particular parameters.

4.1 The effect of risk aversion and habits

Let the initial priors $(\overline{\mu}_i)_{i=1,\dots,N}$ and the over-confidence parameters $(\phi_i)_{i=1,\dots,N}$ be fixed and identical for all agents. As will be seen in the proof of Theorem 4.2, the survival index is invariant under additive translation, and thus it is determined in the current setting by

$$\rho_i + \left(\mu - \frac{(\sigma^D)^2}{2}\right)\left(\gamma_i + (1-\gamma_i)\beta_i\right).$$

If $\beta_1 = \cdots = \beta_N = 0$, the survival index is the same as in Cvitanić et al. [11]. In particular, in a growing economy (i.e., $\mu - (\sigma^D)^2/2 > 0$), the least risk-averse agent will survive in the long run, as in the models of Yan [32], and Cvitanić et al. [11]. The presence of habits may change the behavior. Here, if the habit is sufficiently strong ($\beta_i > 1$), the effect completely reverses: It is the most risk-averse agent who survives in the long run. Effectively, 'catching up with the Joneses' preferences change an agent's risk aversion from γ_i to

$$\gamma_i + (1-\gamma_i)\beta_i. \tag{4.1}$$

Therefore, for strong habits, agents with a high risk aversion effectively behave as agents with a low risk aversion. When risk aversion is homogeneous, the effect of habits' strength on survival depends on whether risk aversion is above or below 1. If risk aversion is above 1, we get the surprising, and at first sight counter-intuitive, result that agents with stronger habits survive in the long run. The reason for this is that the presence of habits forces the agent to trade more aggressively and make bets on very good realizations of the dividend in order to sustain the aggregate habit level generated by the 'catching up with the Joneses' preferences. This makes an agent with strong habits effectively less risk averse. This is beneficial for survival in a growing economy.

4.2 The effect of diverse beliefs

Consider an economy where agents may differ only with respect to their average mean growth rate estimations $(\overline{\mu}_i)_{i=1,\dots,N}$ and their correlation parameters $(\phi_i)_{i=1,\dots,N}$. In this case, the survival index admits the form

$$\kappa_i = \frac{1}{2}\left(\frac{\overline{\mu}_i - \overline{\mu}}{\sigma^D}\right)^2 + \frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}}.$$

Note that in this case the survival index is a decreasing function of the correlation parameter ϕ_i in the interval $[-1, \phi]$, and an increasing function in the interval $(\phi, 1]$. Therefore, in an economy where the only distinction between agents comes from their correlation parameters, the surviving agent is derived as follows. If either all agents are over-confident ($\phi < \phi_i$, for all $i = 1, \dots, N$) or under-confident ($\phi > \phi_i$, for all $i = 1, \dots, N$), then the survival index is given by

$$|\phi_i - \phi|,$$

and thus the individual with the most accurate guess of the correct correlation will dominate the market. If some agents are over-confident and some are under-confident in the signal, the situation becomes more complex. For simplicity, let us analyze the case of an economy which consists of two agents: the first agent underestimates the correlation and believes that it is $\phi_1 \in [-1, \phi]$, whereas the second agent overestimates the correlation by $\phi_2 \in [\phi, 1]$. Let us set $a := t(\frac{\xi\sigma^D}{\sigma^\mu})^2$. If $\phi_1 \in [-1, 2\frac{a\phi(1+a)}{a\phi^2+(a+1-\phi)^2} - 1]$, the second agent will survive. Now, assume that $\phi_1 \in [2\frac{a\phi(1+a)}{a\phi^2+(a+1-\phi)^2} - 1, \phi]$. Then if $\phi_2 \in [\phi, \frac{2(a+1)\phi-(a+1+\phi^2)\phi_1}{a+1+\phi^2-2\phi\phi_1}]$, the second agent will survive; otherwise, namely if $\phi_2 \in [\frac{2(a+1)\phi-(a+1+\phi^2)\phi_1}{a+1+\phi^2-2\phi\phi_1}, 1]$, the first agent will survive. To demonstrate the above scheme numerically, let us consider the case where $a = 1$ and $\phi = 1/2$ (see Fig. 1). If $\phi_1 \in [-1, -0.2]$, then the second agent will survive. If $\phi_1 \in [-0.2, 0.5]$, then, if $\phi_2 \in [0.5, \frac{8-9\phi_1}{9-4\phi_1}]$, the second agent is the one to survive. Otherwise, if $\phi_2 \in [\frac{8-9\phi_1}{9-4\phi_1}, 1]$, then the first agent will survive. The preceding fact yields an economically surprising observation: too over-confident agents will not survive when they compete with agents that believe in a weak negative correlation. Assume for instance that the second agent believes that the correlation is some

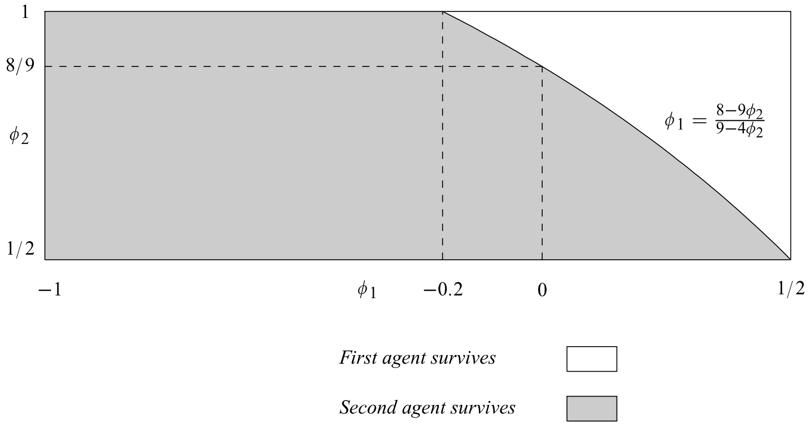

Fig. 1 The long-run surviving consumer

$\phi_2 \in [8/9, 1]$. Then, if $\phi_1 \in [\frac{8-9\phi_2}{9-4\phi_2}, 0]$, the first agent will survive, despite the negative correlation. This is very surprising, since irrational agents who believe in a non-positive correlation happen to survive, whereas individuals with an overestimation of the signal will become extinct.

If the only source of heterogeneity in the economy is the belief regarding the average mean growth rate, then the survival index depends only on the error between the subjective mean growth rate and the correct one, namely,

$$\kappa_i = |\overline{\mu} - \overline{\mu_i}|.$$

Therefore, the consumer with the best forecast of the average mean growth rate is the one to dominate the market.

4.3 The relative level of absolute risk tolerance

As in Cvitanić et al. [11], we define the *relative level of absolute risk tolerance* of agent i by

$$w_{it} := \frac{c_{it}/\gamma_i}{\sum_{j=1}^{N} c_{jt}/\gamma_j}.$$

The following is an immediate consequence of Theorem 4.2.

Corollary 4.3 *We have*

$$\lim_{t\to\infty} w_{it} = 0$$

for all $i \neq I_k$, *and*

$$\lim_{t\to\infty} w_{I_K t} = 1.$$

Proof Note that (3.2) implies that

$$w_{it} = \frac{c_{it}}{D_t} \frac{1/\gamma_i}{\sum_{j=1}^{N} c_{0j}/\gamma_j (M_{jt}/M_t)^{1/\gamma_j}}.$$

The identity (3.3) yields

$$\frac{1}{\sum_{j=1}^{N} \frac{1}{\gamma_j} c_{0j} (M_{jt}/M_t)^{1/\gamma_j}} \le \max_{k=1,\dots,N} \gamma_k.$$

The preceding observations combined with Theorem 4.2 and the equality $\sum_{i=1}^{N} \omega_{it} = 1$ complete the proof of Corollary 4.3. □

5 Auxiliary results

In the present section, we provide some results that will be crucial for proving Theorem 4.2. First, we introduce the following estimates, indicating that y_{it}, $1/y_{it}$, their derivatives and ν_{it} are close to certain functions of a simpler form. The errors in these estimates are shown to be decaying exponentially fast to 0 as $t \to \infty$.

Lemma 5.1 *We have*

$$\left|\nu_{it} - \alpha_{i2}\left(\sigma^D\right)^2\right| \le C e^{-2(\alpha_{i2}+\xi)t}, \tag{5.1}$$

$$\left| y_{it} - \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right) e^{(\alpha_{i2}+\xi)t} \right| \le C e^{-(\alpha_{i2}+\xi)t}, \tag{5.2}$$

$$\left| y'_{it} - (\alpha_{i2}+\xi) \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right) e^{(\alpha_{i2}+\xi)t} \right| \le C e^{-(\alpha_{i2}+\xi)t} \tag{5.3}$$

and

$$\left| \frac{1}{y_{it}} - \exp\left(\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right) e^{-(\alpha_{i2}+\xi)t} \right| \le C e^{-3(\alpha_{i2}+\xi)t}, \tag{5.4}$$

$$\left| \left(\frac{1}{y_{it}}\right)' + (\alpha_{i2}+\xi) \exp\left(\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right) e^{-(\alpha_{i2}+\xi)t} \right| \le C e^{-3(\alpha_{i2}+\xi)t}, \tag{5.5}$$

for all $t > 0$ *and some constant* $C > 0$.

Proof Inequality (5.1) is due to the fact that

$$\left|\nu_{it} - \alpha_{i2}\left(\sigma^D\right)^2\right| = \left| \frac{(\alpha_{i1} - \alpha_{i2})\alpha_{i2}(\sigma^D)^2}{\alpha_{i1} e^{2(\alpha_{i2}+\xi)t} - \alpha_{i2}} \right|.$$

Next, by definition (see Proposition 2.4), it follows that y_{it} admits the form

$$y_{it} = \exp\left((\alpha_{i2}+\xi)t - \frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}} \left(1 - e^{-2(\alpha_{i2}+\xi)t}\right) \right).$$

One checks that the inequality $e^x - 1 \le (e-1)x$, for all $0 \le x \le 1$, concludes the validity of (5.2). Recall that y_i satisfies the ODE $y'_{it} = (\xi + \frac{\nu_{it}}{(\sigma^D)^2}) y_{it}$, and thus we can estimate

$$\begin{aligned}
&\left| y'_{it} - (\alpha_{i2} + \xi) \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}} \right) e^{(\alpha_{i2}+\xi)t} \right| \\
&\quad \le \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}} \right) e^{(\alpha_{i2}+\xi)t} \left| \frac{\nu_{it}}{(\sigma^D)^2} - \alpha_{i2} \right| \\
&\qquad + \left(\xi + \frac{\nu_{it}}{(\sigma^D)^2} \right) \left| y_{it} - \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}} e^{-\frac{\alpha_{i2}}{\alpha_{i1}}} \right) e^{(\alpha_{i2}+\xi)t} \right|,
\end{aligned}$$

which implies (5.3) by applying (5.1) and (5.2). The inequalities (5.4) and (5.5) are proved in a similar manner. □

For $d \ge 1$, we denote by $(C_0([0,1]; \mathbb{R}^d), \|\cdot\|_\infty)$ the space of all $\mathbb{R}^d$-valued continuous functions on the interval $[0,1]$ vanishing at 0, endowed with the sup topology.

Definition 5.2 We denote by $K^{(d)}$ the space of all functions $f = (f_1, \dots, f_d)$ in $C_0([0,1]; \mathbb{R}^d)$ such that each component f_i is absolutely continuous and

$$\sum_{i=1}^d \int_0^T \left(f'_i(x) \right)^2 dx \le 1.$$

We note that $K^{(d)}$ is a compact subset of $C_0([0,1]; \mathbb{R}^d)$ (see Proposition VIII.2.7 in Revuz and Yor [26]). The next result deals with the asymptotics of certain multiple stochastic integrals.

Lemma 5.3 *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two arbitrary standard Brownian motions and denote $Z_t = \int_0^t e^{-s} W_{\frac{1}{2}(e^{2s}-1)}\, dB_s$. Then we have*

(i)

$$\langle Z \rangle_\infty := \lim_{t\to\infty} \langle Z \rangle_t = \infty.$$

(ii)

$$\lim_{t\to\infty} \frac{\int_0^t e^{-as} \int_0^s e^{au}\, dW_u\, dB_s}{t} = 0 \quad \textit{for any } a > 0.$$

(iii)

$$\lim_{t\to\infty} \frac{\int_0^t e^{-(a+b)s} \int_0^s e^{au} \int_0^u e^{bx}\, dW_x\, du\, dB_s}{t} = 0 \quad \textit{for all } a, b > 0.$$

Proof (i) First, note that a change of variable implies that

$$\langle Z \rangle_t = \int_0^t e^{-2s} (W_{\frac{1}{2}(e^{2s}-1)})^2\, ds = \int_0^{\frac{1}{2}(e^{2t}-1)} \frac{W_u^2}{(1+u)^2}\, du.$$

Consider the functional $F : C_0([0,1];\mathbb{R}) \to \mathbb{R}_+$ given by

$$F(f) := \int_0^1 \frac{f^2(x)}{(1+x)^2}\,dx.$$

Note that F is continuous. Indeed, for a fixed $f \in C_0([0,1];\mathbb{R})$ and all $\varepsilon > 0$, let $\delta = \varepsilon(2\|f\|_\infty + \varepsilon)$ and observe that $\|f-g\|_\infty < \delta$ for some $g \in C_0([0,1];\mathbb{R})$ implies $|F(f) - F(g)| < \varepsilon$. It follows by Strassen's functional law of the iterated logarithm (see Theorem VIII.2.12 in Revuz and Yor [26]) that P-a.s.,

$$\limsup_{N\to\infty} F\left(\frac{1}{\sqrt{2N\log\log N}} W_{Nt}\right) = \sup_{h\in K^{(1)}} F(h).$$

Notice that $\sup_{h\in K^{(1)}} F(h) \geq F(\tilde{h}) > 0$, where $\tilde{h}(x) = x$. Therefore, we have

$$\begin{aligned}\limsup_{N\to\infty} F\left(\frac{1}{\sqrt{2N\log\log N}} W_{Nt}\right) &= \limsup_{N\to\infty} \frac{\int_0^1 \frac{W_{Nt}}{(1+t)^2}\,dt}{2N\log\log N}\\ &= \limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(N+u)^2}\,du}{2\log\log N} > 0.\end{aligned}$$

Furthermore,

$$\limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(1+u)^2}\,du}{2\log\log N} \geq \limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(N+u)^2}\,du}{2\log\log N} > 0.$$

In particular, it follows that

$$\limsup_{N\to\infty} \int_0^N \frac{W_u^2}{(1+u)^2}\,du = \infty,$$

but since $N \mapsto \int_0^N \frac{W_u^2}{(1+u)^2}\,du$ is increasing, we get $\lim_{N\to\infty} \int_0^N \frac{W_u^2}{(1+u)^2}\,du = \infty$. This accomplishes the proof of part (i).

(ii) Denote $Y_t = \int_0^t e^{-s} \int_0^s e^u\,dW_u\,dB_s$ and $X_s = \int_0^s e^u\,dW_u$. Note that we have $\langle X\rangle_t = \frac{1}{2}(e^{2t}-1)$. Since X is a martingale null at 0 and $\langle X\rangle_\infty = \infty$, it follows by the Dambis–Dubins–Schwarz theorem (shortly DDS, see Theorem V.1.6 in Revuz and Yor [26]) that $X_t = \widetilde{W}_{\frac{1}{2}(e^{2t}-1)}$ for some Brownian motion $\widetilde{W}$. Therefore, we can rewrite

$$Y_t = \int_0^t e^{-s}\widetilde{W}_{\frac{1}{2}(e^{2s}-1)}\,dB_s,$$

and thus by part (i), we have $\lim_{t\to\infty}\langle Y\rangle_t = \langle Y\rangle_\infty = \infty$. It follows from the DDS theorem that $Y_t = \widetilde{B}_{\langle Y\rangle_t}$, for some Brownian motion $\widetilde{B}$. Now, denote $\phi(x) = \sqrt{2x\log\log x}$ and rewrite $\frac{Y_t}{t} = \frac{\widetilde{B}_{\langle Y\rangle_t}}{\phi(\langle Y\rangle_t)}\frac{\phi(\langle Y\rangle_t)}{t}$. By the law of the iterated logarithm, we have that $\limsup_{t\to\infty}\frac{|\widetilde{B}_{\langle Y\rangle_t}|}{\phi(\langle Y\rangle_t)} \leq 1$, and hence it is enough to concentrate on the asymptotics of the second term

$$\frac{\phi(\langle Y\rangle_t)}{t}=\sqrt{2\frac{\int_0^t e^{-2s}(\widetilde{W}_{\frac{1}{2}(e^{2t}-1)})^2\,ds\log\log(\int_0^t e^{-2s}(\widetilde{W}_{\frac{1}{2}(e^{2t}-1)})^2\,ds)}{t^2}}.$$

Note that $\phi(\frac{1}{2}(e^{2s}-1))\le e^s\sqrt{\log 2s}$ and thus, the law of the iterated logarithm implies that

$$\limsup_{t\to\infty}\frac{\phi(\langle Y\rangle_t)}{t}\le\limsup_{t\to\infty}\sqrt{\frac{\log(2t)\log\log(t\log 2t)}{t}}=0.$$

This accomplishes the proof of part (ii).

(iii) By Fubini's theorem, we have

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{au}\int_0^u e^{bx}\,dW_x\,du\,dB_s}{t}$$
$$=\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{bx}\,dW_x\,dB_s}{t}-\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{(a+b)x}\,dW_x\,dB_s}{t}=0,$$

where the last equality follows by part (ii). This completes the proof of Lemma 5.3. □

We proceed with the following statement.

Lemma 5.4 *Let $(W_t)_{t\in[0,\infty)}$ be a standard Brownian motion. Then we have*

(i)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{ax}\,dW_x\,ds}{t}=0\quad\textit{for all } a>0.$$

(ii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{au}\int_0^u e^{bx}\,dW_x\,du\,ds}{t}=0\quad\textit{for all } a,b>0.$$

Proof (i) By using integration by parts and Fubini's theorem, we get

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{au}\,dW_u\,ds}{t}=\lim_{t\to\infty}\frac{\int_0^t(W_s-ae^{-as}\int_0^s e^{au}W_u\,du)\,ds}{t}$$
$$=\lim_{t\to\infty}\frac{\int_0^t W_s\,ds-a\int_0^t(e^{au}W_u\int_u^t e^{-as}\,ds)\,du}{t}=\lim_{t\to\infty}\frac{\int_0^t e^{au}W_u\,du}{te^{at}}=0,$$

where the last equality follows by the law of large numbers.

(ii) As in (i), one checks that the limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\left(\int_0^t e^{-bs}\int_0^s e^{bx}\,dW_x\,ds-\int_0^t e^{-(a+b)s}\int_0^s e^{(a+b)x}\,dW_x\,ds\right),$$

which vanishes according to (i). □

In the next limit theorems, the main tool is ergodicity of certain stochastic processes. Similar ideas as below (even though we have provided a direct argument) could be applicable to deduce the previous lemma.

Lemma 5.5 *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two independent Brownian motions. Then the following hold*:

(i)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{ax}\,dW_x e^{-bs}\int_0^s e^{bx}\,dB_x\,ds}{t}=0\quad \textit{for all } a,b>0.$$

(ii)

$$\lim_{t\to\infty}\frac{\int_0^t (e^{-as}\int_0^s e^{ax}\,dW_x)^2\,ds}{t}=\frac{1}{2a}\quad \textit{for all } a>0.$$

(iii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{ax}\,dW_x\int_0^s e^{bx}\,dW_x\,ds}{t}=\frac{1}{a+b}\quad \textit{for all } a,b>0.$$

Proof (i) First observe that $\int_0^\cdot e^{ax}\,dW_x$ is a martingale with $\langle\int_0^\cdot e^{ax}\,dW_x\rangle_t=\frac{e^{2at}-1}{2a}$, and thus by the DDS theorem, we have $\int_0^t e^{ax}\,dW_x=\widetilde{W}_{\frac{e^{2at}-1}{2a}}$ for some Brownian motion $(\widetilde{W}_t)_{t\in[0,\infty)}$. A similar argument implies that $\int_0^t e^{bx}\,dB_x=\widetilde{B}_{\frac{e^{2bt}-1}{2b}}$ for a Brownian motion $(\widetilde{B}_t)_{t\in[0,\infty)}$. The construction in the DDS theorem implies that $(\widetilde{B}_t)_{t\in[0,\infty)}$ and $(\widetilde{B}_t)_{t\in[0,\infty)}$ are independent. Recall that $(e^{-at}\widetilde{W}_{e^{2at}})$ and $(e^{-bt}\widetilde{B}_{e^{2bt}})$ are two independent stationary Ornstein–Uhlenbeck processes; thus the process $(e^{-(a+b)t}\widetilde{W}_{e^{2at}}\widetilde{B}_{e^{2bt}})$ is stationary. Therefore, an ergodic theorem for stationary processes implies that

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\widetilde{W}_{e^{2as}}\widetilde{B}_{e^{2bs}}\,ds}{t}=e^{-(a+b)}E\big[\widetilde{W}_{e^{2a}}\widetilde{B}_{e^{2b}}\big]=0. \tag{5.6}$$

Next, the process $(W'_t)_{t\in[0,\infty)}$ given by

$$W'_t=\begin{cases}\sqrt{2a}\,\widetilde{W}_{\frac{t}{2a}} & \text{for } t<1,\\ \sqrt{2a}\,\widetilde{W}_{\frac{t-1}{2a}}+\sqrt{2a}\,\widetilde{W}_{\frac{1}{2a}} & \text{for } t\ge 1\end{cases}$$

is a Brownian motion. Thus, we have $\widetilde{W}_{\frac{e^{2as}-1}{2a}}=\frac{1}{\sqrt{2a}}W'_{e^{2as}}-\widetilde{W}_{\frac{1}{2a}}$ for all $s>1$. We define the process $(B'_t)_{t\in[0,\infty)}$ in a similar manner. We emphasize that $(W'_t)_{t\in[0,\infty)}$ and $(\widetilde{W}_t)_{t\in[0,\infty)}$ are independent of $(B'_t)_{t\in[0,\infty)}$ and $(\widetilde{B}_t)_{t\in[0,\infty)}$. Thus we can rewrite (5.6) as

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}(\frac{1}{\sqrt{2a}}W'_{e^{2as}}-\widetilde{W}_{\frac{1}{2a}})(\frac{1}{\sqrt{2b}}B'_{e^{2bs}}-\widetilde{B}_{\frac{1}{2a}})\,ds}{t}.$$

Next, the law of the iterated logarithm implies that for every $\varepsilon > 0$, there exists an $\mathcal{F}_\infty$-measurable random variable $N(\varepsilon) : \Omega \to \mathbb{R}_+$ such that for all $s > N(\varepsilon)$, we have $|\frac{W_{e^{2as}}}{e^{as}\sqrt{\log(2as)}}| < 1 + \varepsilon$, and hence

$$\lim_{t\to\infty} \frac{\int_0^t |e^{-as-bs} W'_{e^{2as}}|\, ds}{t} \le (1+\varepsilon) \lim_{t\to\infty} \frac{\int_0^t \frac{\log(as)}{e^{bs}}\, ds}{t} = 0.$$

This fact combined with (5.6) accomplishes the proof of part (i).

(ii) As in (i), $\int_0^s e^{ax}\, dW_x = \widetilde{W}_{\frac{e^{2as}-1}{2a}}$ and $\widetilde{W}_{\frac{e^{2as}-1}{2a}} = \frac{1}{\sqrt{2a}} W'_{e^{2as}} - \widetilde{W}_{\frac{1}{2a}}$. Next, ergodicity yields

$$\lim_{t\to\infty} \frac{\int_0^t (e^{-as} \widetilde{W}_{e^{2as}})^2\, ds}{t} = \frac{1}{e^{2a}} E\big[\widetilde{W}^2_{e^{2a}}\big] = 1.$$

Finally, the above limit combined with similar arguments to those appearing in (i) concludes the proof.

(iii) The idea is to rewrite the required limit in terms of limits of the same form as those in (ii). First, observe that $e^{-at} \int_0^s e^{au}\, dW_u = W_s - ae^{-at} \int_0^s e^{au} W_u\, du$. Thus we can rewrite

$$\begin{aligned}\int_0^t \left(e^{-as} \int_0^s e^{au}\, dW_u\right)^2 ds &= \int_0^t W_s^2\, ds - 2a \int_0^t e^{-as} W_s \int_0^s e^{au} W_u\, du\, ds \\ &\quad + a^2 \int_0^t e^{-2as} \left(\int_0^s e^{au} W_u\, du\right)^2 du. \end{aligned} \tag{5.7}$$

Observe that Fubini's theorem implies that

$$\begin{aligned}\int_0^t e^{-2as} \left(\int_0^s e^{au} W_u\, du\right)^2 du &= \int_0^t \int_0^t e^{ax+ay} W_x W_y \int_{\max\{x,y\}}^t e^{-2as}\, ds\, dx\, dy \\ = \frac{1}{a} \int_0^t e^{-as} W_s \int_0^s e^{au} W_u\, du\, ds &- \frac{1}{2ae^{2at}} \left(\int_0^t W_x e^{ax}\, dx\right)^2. \end{aligned} \tag{5.8}$$

This fact alongside (5.7) and (5.8) implies that

$$\begin{aligned}&\lim_{t\to\infty} \frac{\int_0^t (e^{-at} \int_0^s e^{au}\, dW_u)^2\, ds}{t} \\ &\quad = \lim_{t\to\infty} \frac{\int_0^t W_s^2\, ds - a \int_0^t e^{-as} W_s \int_0^s e^{au} W_u\, du\, ds - \frac{a}{2e^{2at}} (\int_0^t e^{as} W_s\, ds)^2}{t}.\end{aligned}$$

By using similar arguments and exploiting the preceding observations, one can check that

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{ax}\,dW_x\int_0^s e^{bx}\,dW_x\,ds}{t}$$
$$=\frac{a}{a+b}\lim_{t\to\infty}\frac{\int_0^t W_s^2\,ds-a\int_0^t e^{-as}W_s\int_0^s e^{au}W_u\,du\,ds+\frac{a}{2e^{2at}}(\int_0^t e^{as}W_s\,ds)^2}{t}$$
$$+\frac{b}{a+b}\lim_{t\to\infty}\frac{\int_0^t W_s^2\,ds-b\int_0^t e^{-as}W_s\int_0^s e^{au}W_u\,du\,ds+\frac{a}{2e^{2at}}(\int_0^t e^{as}W_s\,ds)^2}{t}.$$

The latter fact combined with part (ii) completes the proof. □

The next statement is heavily based on the previous lemma.

Lemma 5.6 *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two independent Brownian motions. Then we have*

(i)

$$\lim_{t\to\infty}\frac{\int_0^t (e^{-(a+b)s}\int_0^s e^{ax}\int_0^x e^{bu}\,dW_u\,dx)^2\,ds}{t}=\frac{1}{2b(a+b)(a+2b)}$$

for all $a,b>0$.

(ii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{au}\,dW_u\int_0^s e^{bu}\int_0^u e^{ax}\,dW_x\,du\,ds}{t}=\frac{1}{2a(2a+b)}$$

for all $a,b>0$.

(iii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{(a-\xi)u}\int_0^u e^{\xi u}\,dW_x\,du\int_0^s e^{(b-\xi)u}\int_0^u e^{\xi u}\,dW_x\,du\,ds}{t}$$
$$=\frac{1}{(a-\xi)(b-\xi)}\left(\frac{1}{a+b}+\frac{1}{2\xi}-\frac{1}{a+\xi}-\frac{1}{b+\xi}\right)\quad \textit{for all } a,b,\xi>0.$$

(iv)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{ay}\int_0^y e^{bu}\,dW_u\,dy\int_0^s e^{(a+b)x}\,dW_x\,ds}{t}=\frac{1}{2(a+b)(a+2b)}$$

for all $a,b>0$.

(v)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{ay}\int_0^y e^{bu}\,dW_u\,dy\int_0^s e^{(a+b)x}\,dB_x\,ds}{t}=0\quad \textit{for all } a,b>0.$$

Proof (i) Notice that $\int_0^s e^{ax}\int_0^x e^{bu}\,dW_u\,dx=\frac{1}{a}\int_0^s e^{bu}(e^{as}-e^{au})\,dW_u$. Therefore, the required limit is equal to

$$\frac{1}{a^2}\lim_{t\to\infty}\frac{\int_0^t (e^{-bs}\int_0^s e^{bu}\,dW_u)^2\,ds}{t} - \frac{2}{a^2}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+2b)s}\int_0^s e^{(a+b)u}\,dW_u\int_0^s e^{bu}\,dW_u\,ds}{t} + \frac{1}{a^2}\lim_{t\to\infty}\frac{\int_0^t (e^{-(a+b)s}\int_0^s e^{(a+b)u}\,dW_u)^2\,ds}{t}.$$

Parts (ii) and (iii) in Lemma 5.5 complete the proof of (i).

(ii) As before, one checks that the limit is equal to

$$\frac{1}{b}\lim_{t\to\infty}\frac{\int_0^t e^{-2as}(\int_0^s e^{ax}dW_x)^2\,ds}{t} - \frac{1}{b}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{au}\,dW_u\int_0^s e^{(a+b)x}\,dW_x\,ds}{t},$$

and the rest is a consequence of parts (ii) and (iii) of Lemma 5.5.

(iii) The limit is equal to

$$\frac{1}{(a-\xi)(b-\xi)}\left(\lim_{t\to\infty}\frac{\int_0^t (e^{-au}\int_0^u e^{ax}\,dW_x)^2\,du + \int_0^t e^{-(a+b)u}\int_0^u e^{ax}\,dW_x\int_0^u e^{bx}\,dW_x\,du}{t} - \lim_{t\to\infty}\frac{\int_0^t e^{-(a+\xi)u}\int_0^u e^{\xi x}\,dW_x\int_0^u e^{ax}\,dW_x\,du + \int_0^t e^{-(b+\xi)u}\int_0^u e^{\xi x}\,dW_x\int_0^u e^{bx}\,dW_x\,du}{t}\right).$$

The rest follows by applying items (ii) and (iii) of Lemma 5.5.

(iv) One checks that the required limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{bu}\,dW_u\int_0^s e^{(a+b)x}\,dW_x\,ds}{t} + \frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}(\int_0^s e^{(a+b)u}\,dW_u)^2 ds}{t},$$

and the rest follows by parts (ii) and (iii) of Lemma 5.5.

(v) As in (i), one checks that the limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{bu}\,dW_u\int_0^s e^{(a+b)x}\,dB_x\,ds}{t} - \frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{(a+b)u}\,dW_u\int_0^s e^{(a+b)x}\,dB_x\,ds}{t},$$

which vanishes due to part (i) of Lemma 5.5. □

6 Proof of the main result

We provide here a proof for Theorem 4.2. Fix an arbitrary $i \neq I_K$. Recall that $\sum_{j=1}^{N} c_{jt} = D_t$, and thus it suffices to show that $\lim_{t\to\infty} \frac{c_{it}}{D_t} = 0$. Note that (3.3) implies that $M_t \geq c_{I_K 0}^{\gamma_{I_K}} M_{I_K t}$. Therefore, (3.2) yields

$$\frac{c_{it}}{D_t} = c_{0i}\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} \leq \frac{c_{i0}}{c_{I_K 0}^{\gamma_{I_K}/\gamma_i}}\left(\frac{M_{it}}{M_{I_K t}}\right)^{1/\gamma_i}.$$

In virtue of (3.1), we have

$$\frac{M_{it}}{M_{I_K t}} = \exp\bigl(a_i(t) - a_{I_K}(t)\bigr),$$

where

$$a_j(t) := (\gamma_j - 1)\beta_j x_t + \left(\frac{(\sigma^D)^2}{2}\gamma_j - \rho_j\right)t + \int_0^t \left(-\gamma_j \mu_s^D - \frac{\delta_{js}^2}{2}\right) ds + \int_0^t \delta_{js}\, dW_s^{(0)} - \gamma_j \sigma^D W_t^{(1)}$$

for all $j = 1, \dots, N$. Therefore, in order to complete the proof of the statement, it suffices to show that

$$\lim_{t\to\infty} \frac{a_i(t) - a_{I_K}(t)}{t} = \kappa_{I_K} - \kappa_i < 0.$$

To this end, we proceed with the computation of the following limits.

Part I. We claim that

$$\lim_{t\to\infty} \frac{x_t}{t} = \overline{\mu} - \frac{1}{2}(\sigma^D)^2. \tag{6.1}$$

Recall that by (2.3) and (2.1), we have

$$\lim_{t\to\infty} \frac{x_t}{t} = \lim_{t\to\infty} \frac{x_0 + \lambda \int_0^t e^{\lambda s}\left(\int_0^s \mu_u^D\, du - \frac{1}{2}(\sigma^D)^2 s + \sigma^D W_s^{(1)}\right) ds}{t e^{\lambda t}}.$$

Note that the law of large numbers implies that $\lim_{t\to\infty} \frac{\int_0^t e^{\lambda s} W_s^{(1)}\, ds}{t e^{\lambda t}} = 0$. Next, it is evident that $\lim_{t\to\infty} \frac{x_0}{t e^{\lambda t}} = 0$ and $\lim_{t\to\infty} \frac{\int_0^t s e^{\lambda s}\, ds}{t e^{\lambda t}} = 1/\lambda$. Let us show that

$$\lim_{t\to\infty} \frac{\int_0^t \mu_u^D\, du}{t} = \overline{\mu}. \tag{6.2}$$

By (2.2), we get

$$\lim_{t\to\infty} \frac{\int_0^t \mu_u^D\, du}{t} = \lim_{t\to\infty} \frac{\int_0^t \left(\overline{\mu} + (\mu_0 - \overline{\mu})e^{-\xi s} + \sigma^\mu \int_0^s e^{\xi(u-s)}\, dW_u^{(2)}\right) ds}{t}.$$

Clearly, we have $\lim_{t\to\infty}\frac{\int_0^t(\overline{\mu}+(\mu_0-\overline{\mu})e^{-\xi s})\,ds}{t}=\overline{\mu}$. Furthermore, part (i) of Lemma 5.4 yields $\lim_{t\to\infty}\frac{\int_0^t\int_0^s e^{\xi(u-s)}\,dW_u^{(2)}\,ds}{t}=0$. This shows the validity of (6.2). Next, by l'Hôpital's rule, we get

$$\lim_{t\to\infty}\frac{\int_0^t e^{\lambda s}\int_0^s \mu_u^D\,du\,ds}{te^{\lambda t}}=\lim_{t\to\infty}\frac{\int_0^t \mu_s^D\,ds}{\lambda t+1}=\frac{\overline{\mu}}{\lambda},$$

proving (6.1).

Part II. We claim that

$$\lim_{t\to\infty}\frac{\int_0^t(\delta_{I_K s}-\delta_{is})\,dW_s^{(1)}}{t}=0.$$

By definition (see (2.12)), it suffices to verify that

$$\lim_{t\to\infty}\frac{\int_0^t \mu_{js}^D\,dW_s^{(1)}}{t}=0$$

holds for all $j=1,\dots,N$. It is not hard to check, by employing Lemma 5.1 combined with the law of large numbers, that the preceding limit does not change when the functions y_{iu}, $\frac{1}{y_{iu}}$ and ν_{iu} are replaced by $e^{(\alpha_{i2}+\xi)t}$, $e^{-(\alpha_{i2}+\xi)t}$ and $\alpha_{i2}(\sigma^D)^2$, respectively. In view of the latter observation, by definition (see (2.8)), we need to show that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\int_0^t((\overline{\mu}_i-\overline{\mu})(1-e^{-\xi s})+(\mu_{0i}-\mu_0)e^{-\xi s}+\frac{\xi\overline{\mu}_i}{\xi+\alpha_{i2}}(1-e^{-(\xi+\alpha_{i2})s}))\,dW_s^{(1)}}{t}\\
&\quad+\alpha_{i2}\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{(\xi+\alpha_{i2})u}(\overline{\mu}+(\mu_0-\overline{\mu})e^{-\xi u})\,du\,dW_s^{(1)}}{t}\\
&\quad+\alpha_{i2}\sigma^\mu\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{\alpha_{i2}u}\int_0^u e^{\xi x}\,dW_x^{(2)}\,du\,dW_s^{(1)}}{t}\\
&\quad+\sigma^\mu\phi_i\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{(\xi+\alpha_{i2})u}\,ds_u\,dW_s^{(1)}}{t}=0.
\end{aligned}$$

One checks that the first two terms vanish by the law of large numbers. The third and fourth limits vanish by parts (iii) and (ii) of Lemma 5.3, respectively. This completes the proof of the second part.

Part III. We have

$$\begin{aligned}
\frac{1}{2}\lim_{t\to\infty}\frac{\int_0^t(\delta_{is}^2-\delta_{I_K s}^2)\,ds}{t}&=\frac{1}{2}\left(\frac{\overline{\mu}_i-\overline{\mu}}{\sigma^D}\right)^2+\frac{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}}\\
&\quad-\frac{1}{2}\left(\frac{\overline{\mu}_{I_K}-\overline{\mu}}{\sigma^D}\right)^2-\frac{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi_{I_K}^2)}}.
\end{aligned}$$

This can be derived by applying Lemmas 5.3–5.6. The proof is now accomplished by combining the above three parts, some routine algebraic transformations and the law of large numbers. □

7 Interest rate and market price of risk: further long-run results

The current section deals with asymptotic results for the interest rate and the market price of risk in heterogeneous economies. More precisely, it is shown that asymptotically, the latter parameters behave as those associated with a homogeneous economy populated by the dominating consumer. Under some mild conditions, we prove that the distance between these parameters in a heterogeneous economy and those associated with *any* of the non-dominating consumer homogeneous economies becomes unbounded as time goes to infinity.

7.1 Market price of risk

The next statement provides a full characterization of the market price of risk asymptotics in heterogeneous economies.

Theorem 7.1 (i) *We have*

$$\lim_{t\to\infty} |\theta_t - \theta_{I_K t}| = 0.$$

(ii) *If $\phi_i = \phi_{I_K}$ for some $i \neq I_K$, then*

$$\lim_{t\to\infty} (\theta_t - \theta_{it}) = \sigma^D(\gamma_{I_K} - \gamma_i) - \frac{1}{\sigma^D}(\overline{\mu}_{I_K} - \overline{\mu}_i).$$

If ϕ_i (for some $i \neq I_K$) is such that

$$\frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}} \neq \frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_{I_K}^2)}},$$

then

$$\limsup_{t\to\infty} |\theta_t - \theta_{it}| = \infty.$$

Proof (i) First, we shall prove that $\lim_{t\to\infty} \omega_{it}\theta_{it} = 0$, for all $i \neq I_K$. As in Sect. 5, the DDS theorem implies the existence of a Brownian motion $(\widetilde{B}(t))_{t\in[0,\infty)}$ such that

$$e^{-at}\int_0^t e^{as}\, dB_s = e^{-at}\widetilde{B}_{\frac{e^{2at}-1}{2a}},$$

where $a > 0$ is some constant and $(B_t)_{t\in[0,\infty)}$ is a Brownian motion. By exploiting the preceding fact, one checks that $\lim_{t\to\infty} \frac{\mu_{it}^D}{\sqrt{\log t}} < \infty$ for all $i = 0, \dots, N$, which implies that

$$\limsup_{t\to\infty} \frac{\theta_{jt}}{\sqrt{\log t}} < \infty \tag{7.1}$$

for $j=1,\dots,N$. On the other hand, Theorem 4.3 shows that $\omega_{it}\le \frac{c_{it}}{D_t}\max_{i=1,\dots,N}\gamma_i$ and all $i\neq I_K$. We have in particular proved in Sect. 6 that $\frac{c_{it}}{D_t}\le e^{-a_i t}$ for some $a_i>0$, for all $i\neq I_K$. This implies that

$$\omega_{it}\le e^{-a_i t}\max_{i=1,\dots,N}\gamma_i \tag{7.2}$$

holds for all $i\neq I_K$, and thus by (7.1), we have $\omega_{it}\theta_{it}\le e^{-a_i' t}$ for all $i\neq I_K$ and some constant $a_i'>0$. Therefore, by Proposition 3.4, we have

$$|\theta_t-\omega_{I_K t}\theta_{I_K t}|=\sum_{i=1,i\neq I_K}^{N}\omega_{it}\theta_{it}\le\sum_{i=1,i\neq I_K}^{N}e^{-a_i' t}. \tag{7.3}$$

Finally, observe that $|\theta_t-\theta_{I_K t}|\le|\theta_t-\omega_{I_K t}\theta_{I_K t}|+\sum_{i=1,i\neq I_K}^{N}\omega_{it}\theta_{I_K t}$ because we have $\sum_{i=1}^{N}\omega_{it}=1$. The proof of part (i) follows from (7.1)–(7.3).

(ii) If $\phi_i=\phi_{I_K}$, by part (i) we can substitute θ_T by $\theta_{I_K T}$. The assertion follows by noting that

$$\lim_{t\to\infty}|\theta_{I_K t}-\theta_{it}|=\lim_{t\to\infty}\left|\sigma^D(\gamma_i-\gamma_{I_K})+\frac{1}{\sigma^D}(\mu_{I_K t}-\mu_{it})\right|.$$

Assume now that

$$\frac{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}}\neq\frac{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2(1-\phi_{I_K}^2)}}.$$

By part (i), the claim is equivalent to proving that

$$\limsup_{t\to\infty}\left|\mu_{I_K t}^D-\mu_{it}^D\right|=\infty. \tag{7.4}$$

First, one checks by employing Lemma 5.1 that the limit in (7.4) does not change when we substitute ν_{it}, y_{it} and $\frac{1}{y_{it}}$ by the terms $\alpha_{i2}(\sigma^D)^2$, $\exp(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}})e^{(\alpha_{i2}+\xi)t}$ and $\exp(\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}})e^{-(\alpha_{i2}+\xi)t}$, respectively. Next, note that Fubini's theorem yields

$$\begin{aligned}&\frac{\alpha_{i2}}{e^{(\xi+\alpha_{i2})T}}\int_0^T e^{\alpha_{i2}u}\int_0^u e^{\xi x}\,dW_x^{(2)}\,du\\&\quad=\frac{1}{e^{\xi T}}\int_0^T e^{\xi u}\,dW_u^{(2)}-\frac{1}{e^{(\alpha_{i2}+\xi)T}}\int_0^T e^{(\alpha_{i2}+\xi)u}\,dW_u^{(2)}.\end{aligned}$$

By exploiting the latter observations and the DDS theorem, one checks that

$$\limsup_{t\to\infty}\left|\mu_{it}^D-\mu_{I_k t}^D\right|=\limsup_{t\to\infty}\left|f_i(t)-f_{I_k}(t)\right|,$$

where

$$f_i(t)=\frac{1}{\sqrt{(\alpha_{i2}+\xi)t}}\left(\sigma^D\alpha_{i2}B_t^{i1}-\sigma^\mu(\phi\phi_i-1)B_t^{i2}+\sigma^\mu\phi_i\sqrt{1-\phi^2}B_t^{i3}\right).$$

Here, B^{i1}, B^{i2} and B^{i3} denote three independent Brownian motions. By applying the DDS theorem again, we can rewrite

$$f_i(t) = \frac{1}{\sqrt{(\alpha_{i2}+\xi)t}} B^{(i)}_{\ell_i t}, \tag{7.5}$$

where $B^{(i)}$ is a Brownian motion, and

$$\ell_i = \left(\sigma^D \alpha_{i2}\right)^2 + \left(\sigma^\mu\right)^2 (1-\phi\phi_i)^2 + \left(\sigma^\mu \phi_i\right)^2 \left(1-\phi^2\right).$$

Lastly, one checks that $\limsup_{t\to\infty} |f_i(t) - f_{I_k}(t)| = \infty$ by using the law of iterated logarithm and (7.5), combined with the fact that

$$\frac{\ell_i}{\alpha_{i2}+\xi} = -2\xi\sigma^D + 2\left(\sigma^D\right)^2 \frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}}.$$

This completes the proof of Theorem 7.1. □

7.2 Interest rate

Analogously to Theorem 7.1, we analyze in the next statement the asymptotics of the interest rate in heterogeneous economies.

Theorem 7.2 (i) *We have*

$$\lim_{t\to\infty} |r_t - r_{I_K t}| = 0.$$

(ii) *If* $\gamma_i = \gamma_{I_K}$, $\beta_i = \beta_{I_K}$ *and* $\phi_i = \phi_{I_K}$ *for some* $i \neq I_K$, *then*

$$\lim_{t\to\infty} (r_t - r_{it}) = \rho_{I_K} - \rho_i + \gamma_{I_K}(\overline{\mu}_{I_K} - \overline{\mu}_i).$$

If at least one of the conditions $\gamma_i = \gamma_{I_K}$, $\beta_i = \beta_{I_K}$ *and*

$$\frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)}} = \frac{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_{I_K}^2)}}$$

does not hold for some $i \neq I_K$, *then*

$$\limsup_{t\to\infty} |r_t - r_{it}| = \infty.$$

Proof (i) By definition, we have

$$r_t - \omega_{it} r_{it} = \sum_{j=1, j\neq I_K}^{N} \omega_{jt} r_{jt} + \frac{1}{2}\sum_{j=1}^{N} (1-1/\gamma_j)\omega_{jt}(\theta_{jt} - \theta_t)^2$$

for all $i=1,\dots,N$. We start by treating the second term. Observe that Theorem 4.3, part (i) of Theorem 7.1, (7.4) and (7.2) imply that

$$\sum_{j=1}^{N}(1-1/\gamma_j)\omega_{jt}(\theta_{jt}-\theta_t)^2\le e^{-a't}$$

for some constant $a'>0$. Next, note that (7.2) yields

$$\sum_{j=1, j\neq I_K}^{N}|\omega_{jt}r_{jt}|\le\sum_{j=1, j\neq I_K}^{N}e^{-a_jt}|r_{jt}|.$$

As in the proof of Theorem 4.2, one can check that $\limsup_{t\to\infty}\frac{r_{jt}}{t}<\infty$ for all $j=1,\dots,N$, and thus we conclude that

$$|r_t-\omega_{I_Kt}r_{I_Kt}|\le e^{-a't}$$

for some constant $a'>0$. Finally, the proof of item (i) is accomplished by employing the inequality $|r_t-r_{I_Kt}|\le|r_t-\omega_{I_Kt}r_{I_Kt}|+r_{I_Kt}|1-\omega_{I_Kt}|$, combined with the fact that $1=\sum_{j=1}^{N}\omega_{jt}$, (7.2) and the fact that $\limsup_{t\to\infty}\frac{r_{jt}}{t}<\infty$ for all $j=1,\dots,N$.

(ii) If $\phi_i=\phi_{I_K}$, $\gamma_i=\gamma_{I_K}$ and $\beta_i=\beta_{I_K}$ for some $i\neq I_K$, the claim follows by combining part (i) with the fact that

$$|r_{it}-r_{I_Kt}|=\big|\rho_{I_K}-\rho_i+\gamma_{I_K}\big(\mu^D_{I_Kt}-\mu^D_{it}\big)\big|.$$

Now, if at least one of the indicated conditions fails for some $i\neq I_K$, the proof is in the same spirit as the one of item (ii) of Theorem 7.1. The only difference is as follows. If $\lambda=\xi$, one can check that the problem can be reduced to proving that

$$\limsup_{t\to\infty}e^{-\lambda t}\left(\sigma^D\int_0^t e^{\lambda u}\,dW_u^{(1)}+\int_0^t\int_0^s e^{\lambda u}\,dW_u^{(2)}\,ds\right)=\infty. \tag{7.6}$$

If $\lambda=0$, we need to prove that

$$\limsup_{t\to\infty}\left(\sigma^D W_t^{(1)}+\int_0^t W_s^{(2)}\,ds\right)=\infty.$$

Let $G:C_0([0,1];\mathbb{R})\to\mathbb{R}$ be given by $G(f)=\int_0^1 f(x)\,dx$. Note that G is continuous, since $|G(f)-G(g)|\le\|f-g\|_\infty$ holds for all $f,g\in C_0([0,1];\mathbb{R})$. By Strassen's functional law of the iterated logarithm, we have

$$\limsup_{N\to\infty}G\left(\frac{1}{\sqrt{2N\log\log N}}W_{Nx}^{(2)}\right)=\limsup_{N\to\infty}\frac{\int_0^N W_u^{(2)}\,du}{N^{3/2}\sqrt{2\log\log N}}=\max_{f\in K^{(1)}}G(f),$$

where $K^{(1)}$ is given in Definition 5.2. Note that $\max_{f\in K^{(1)}}G(f)\ge G(f_0)>0$, where $f_0(x)=x$. Combining the fact they $\lim_{t\to\infty}\frac{W_t^{(1)}}{t^{3/2}\sqrt{\log\log t}}=0$ with the preceding observation shows that (7.6) holds for $\lambda=0$. Assume next that $\lambda\neq 0$. By the DDS theorem, (7.6) is equivalent to

$$\limsup_{t\to\infty} e^{-\lambda t}\left(\sigma^D B^{(1)}\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)+\int_0^t B^{(2)}\left(\frac{e^{2\lambda s}-1}{2\lambda}\right)ds\right)=\infty,$$

where $B^{(1)}$ and $B^{(2)}$ denote two standard independent Brownian motions. By a change of variables, the claim is equivalent to

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\left(\sigma^D B^{(1)}(t)+\int_0^t \frac{B^{(2)}(u)}{1+2\lambda u}\,du\right)=\infty. \tag{7.7}$$

The law of the iterated logarithm yields $\lim_{t\to\infty}\frac{\int_1^t \frac{B^{(1)}(u)}{u(1+2\lambda u)}\,du}{\sqrt{t}}=0$, and thus (7.7) can be rewritten as

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\left(\sigma^D B^{(1)}(t)+\frac{1}{2\lambda}\int_1^t \frac{B^{(2)}(u)}{u}\,du\right)=\infty.$$

Fix some $0<\varepsilon<1$. Consider the functional $H: C_0([0,1];\mathbb{R}^2)\to\mathbb{R}$, given by

$$H(f,g):=\sigma^D f(1)+\frac{1}{2\lambda}\int_\varepsilon^1 \frac{g(u)}{u}\,du.$$

Note that H is continuous, since

$$\left|H(f,g)-H(\widehat{f},\widehat{g})\right|\le \sigma^D\|f-\widehat{f}\|_\infty-\frac{\log\varepsilon}{2\lambda}\|g-\widehat{g}\|_\infty$$

is satisfied for all $f,g,\widehat{f},\widehat{g}\in C_0([0,1];\mathbb{R}^2)$. Next, Strassen's functional law of the iterated logarithm yields

$$\limsup_{N\to\infty} H\left(\frac{1}{\sqrt{2N\log\log N}}\left(B^{(1)}(Nt),B^{(2)}(Nt)\right)\right)=\max_{(f,g)\in K^{(2)}} H(f,g),$$

where $K^{(2)}$ is introduced in Definition 5.2. Observe that

$$\max_{(f,g)\in K^{(2)}} H(f,g)\ge H(h,h)>0,$$

where $h(x)=x$. Therefore, we obtain that

$$\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_{\varepsilon N}^N \frac{B^{(2)}(u)}{u}\,du\right)>0. \tag{7.8}$$

We claim next that

$$\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N \frac{B^{(2)}(u)}{u}\,du\right)>0.$$

Assume towards contradiction that this is not the case. Then, Kolmogorov's 0-1 law implies that

$$P\left(\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}\,du\right)>0\right)=0.$$

Therefore, by exploiting the symmetry of Brownian motion, we obtain that

$$\lim_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}\,du\right)=0$$

holds P-a.s. But since σ^D and λ were arbitrary, we obtain that

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_{\varepsilon N}^N\frac{B^{(2)}(u)}{u}\,du\right)\\
&\quad=\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\left(\sigma^{(D)}-\sqrt{\varepsilon}\right)B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}\,du\right.\\
&\qquad\left.+\widetilde{B}^{(1)}(\varepsilon N)-\frac{1}{2\lambda}\int_1^{\varepsilon N}\frac{B^{(2)}(u)}{u}\,du\right)=0,
\end{aligned}$$

where $\widetilde{B}^{(1)}(t)=\sqrt{\varepsilon}B^{(1)}(\frac{t}{\varepsilon})$, $t\geq 0$, is a Brownian motion (independent of $B^{(2)}$), and $\varepsilon>0$ is sufficiently small. This is a contradiction to (7.8), proving (7.7). □

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